FORCE AND MOMENT ON A SLENDER BODY OF
REVOLUTION MOVING IN WATER OF FINITE DEPTH

by

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ABSTRACT

A consistent first-order theory is derived for the heave force and pitching moment on an arbitrary slender body of revolution moving horizontally with constant velocity in water of finite depth. A computer program is given for evaluating the resulting expression. Numerical results obtained using this program are presented and compared with previous theory and with published experimental results for the case of a Rankine ovoid.

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1. INTRODUCTION

When a submerged body moves horizontally near a free surface, the body experiences a vertical force and a pitching moment due to the presence of the free surface. If the water is shallow, there will also be a force and moment due to the presence of the bottom. In addition, in this case the free surface waves, and hence the force and moment due to the presence of the free surface, will be altered. The magnitude and sign of the force and moment will depend in a complicated manner on the geometry, location and Froude number of the body, and on the depth of water.

In order to obtain a tractable problem, some simplifying assumptions must be made. The flow is considered to be inviscid, incompressible, homogeneous, and irrotational. The analysis is carried out for a slender body of revolution, which can be represented by a line distribution of sources along the body axis, as is usual for a slender body, and a similar distribution of vertical dipoles to correct for the crossflow induced by the presence of the free surface and bottom.

Although the source distribution is of lower order than the dipole distribution, both result in contributions to the moment of equal order. The expression for the potential is also obtained directly for the far field by applying Green's theorem, which justifies the assumed form of the potential. Numerical results are presented for the case of a Rankine ovoid to demonstrate the effect of depth of submergence, depth of water, and Froude number. These results are compared with published experimental results for the case of infinite depth of water.

This problem has been studied previously by Havelock [1] for the special case of a prolate spheroid in deep water. He justifies the second (dipole) term in the expression for the moment on order of magnitude grounds and suggests its applicability to other bodies. Pond [2] , who has studied the case of a Rankine ovoid moving in deep water, derives the dipole term by a heuristic argument and justifies its use by comparison with experimental results. The other limiting case has been studied by Newman [3], who calculated the force and moment on a slender body of revolution moving laterally near a wall, but with no free surface.

2. FORMULATION OF THE PROBLEM FOR THE POTENTIAL

We assume a slender body of revolution to be moving horizontally with a constant velocity U parallel to its axis at a depth b beneath the free surface of a body of water of uniform depth h. Let $Oxyz$ be a Cartesian coordinate system with the z-axis upward moving with velocity U in the direction Ox parallel to the body axis. The origin is located on the free surface directly above the midpoint of the body axis. This is shown in Figure 1. We nondimensionalize all quantities with respect to body length L, the velocity U, and the fluid density ρ.

With the further assumption of irrotational flow in an inviscid, incompressible, and homogeneous fluid, there exists a potential Φ such that the velocity components (u,v,w) are given by

$$u = \Phi_x, \qquad v = \Phi_y, \qquad w = \Phi_z \tag{1}$$

From the continuity equation and the assumption of incompressibility, we obtain Laplace's equation

$$\nabla^2 \Phi = \Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0 \tag{2}$$

On the free surface, which is at an unknown location which is to be found as a part of the solution of the problem, the dynamic boundary condition that the pressure be constant must be satisfied. Thus from Bernoulli's equation,

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}|\nabla \Phi|^2 + \frac{p}{\rho} + gz = C(t) \tag{3}$$

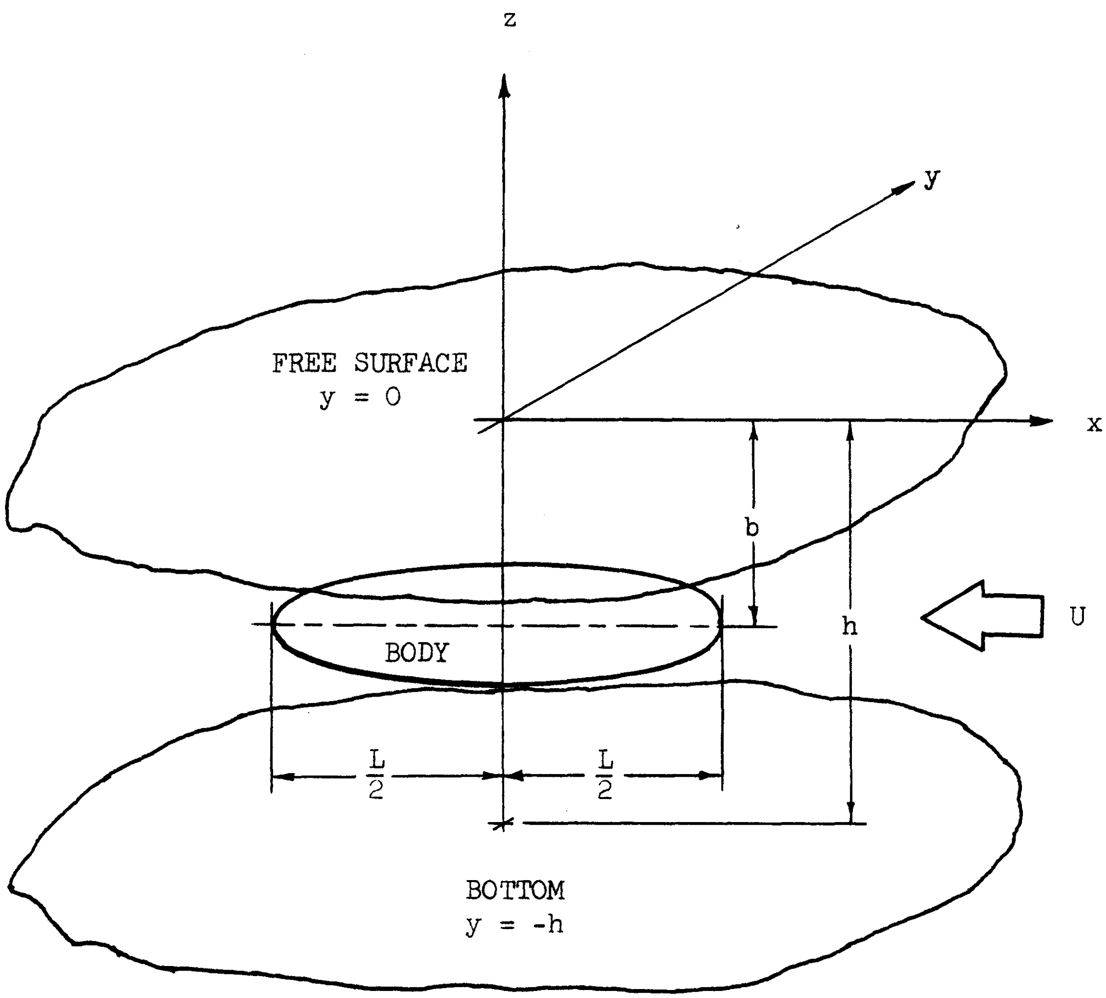


Figure 1 The Coordinate System and Flow Geometry

where p is the pressure, g is the gravitational acceleration, and $C(t)$ is a function of time only, which we may take as identically zero. On the free surface we must also satisfy the kinematic boundary condition, which requires that a particle of fluid on the surface remains on the surface; that is, has the same velocity normal to the surface as the surface itself has. Thus if the free surface is given by

$$F(x, y, z, t) = z - \zeta(x, y, t) = 0 \tag{4}$$

we must have

$$\frac{DF}{Dt} = 0 \tag{5}$$

on the surface, where D/Dt is the substantial derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{6}$$

On any solid boundary we require that the velocity be tangent to the surface, or

$$\frac{\partial \Phi}{\partial n} = 0 \tag{7}$$

on the surface, where $\partial/\partial n$ is the derivative normal to the surface. In the case of a horizontal bottom, this is just

$$\frac{\partial \Phi}{\partial z} = 0 \qquad \text{on} \quad z = -h \tag{8}$$

The fluid velocities must be bounded everywhere in the fluid. Finally, to insure uniqueness of the solution, we must impose a radiation condition at infinity. If the problem were formulated as the limit for large time of an initial value problem, this difficulty would not arise

(Stoker, [4]). However, it is simpler to avoid the time dependence by solving a steady-state problem with the appropriate additional condition. Thus we require that ahead of the body the motion must vanish, except for the uniform stream,

$$\lim_{x \to \infty} \nabla \Phi = (-U, 0, 0) + o(x^{-1/2}) \qquad (9)$$

Since we want the potential for a body in a uniform stream, it is natural to write

$$\Phi = \phi - Ux \qquad (10)$$

where ϕ is the disturbance potential due to the presence of the body. Then in terms of ϕ , we obtain the problem

$$\nabla^2 \phi = 0 \qquad (11)$$

$$\frac{\partial \phi}{\partial z} + \frac{U^2}{g} \frac{\partial^2 \phi}{\partial x^2} = \frac{U}{g} \frac{\partial}{\partial x} |\nabla \phi|^2 - \frac{1}{2g}\left(\frac{\partial \phi}{\partial x}\frac{\partial}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial y} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial z}\right) |\nabla \phi|^2 \qquad (12)$$

on $z = \zeta(x, z)$ from equations (3) and (5) for the time-independent case,

$$\frac{\partial \phi}{\partial n} = U \frac{\partial x}{\partial n} \qquad (13)$$

on the body surface,

$$\frac{\partial \phi}{\partial z} = 0 \tag{14}$$

on $z = -h$, and the radiation condition

$$\lim_{x \to \infty} \nabla \phi = o(x^{-1/2}) \tag{15}$$

We also require that ϕ be bounded at infinity. The bottom condition for $h \to \infty$ is

$$\lim_{z \to -\infty} \nabla \phi = 0 \tag{16}$$

The free surface condition (12) is nonlinear and must be evaluated at an unknown location. By expanding the potential in a Taylor series about $z = 0$ and taking the lowest order terms in ζ, we obtain the linearized free surface condition

$$\frac{\partial \phi}{\partial z} + \frac{U^2}{g} \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{17}$$

to be satisfied on the undisturbed free surface $z = 0$ (cf. Weyhause and Laitone, [5]). The wave height is, to lowest order,

$$\zeta(x, y) = \frac{U}{g} \frac{\partial \phi}{\partial x} \tag{18}$$

and thus we have neglected terms of order $O(\phi^2)$ in the boundary condition.

3. SOLUTION FOR THE POTENTIAL

We now seek an approximate solution to the linearized free surface problem formulated in the preceding section. The solution will be obtained using a slender-body approximation. We first consider a line distribution of sources of arbitrary strength $\sigma(x)$ on the body axis over the interval $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

$$
\begin{aligned}
I_{11}(x,y,z) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma(\xi)\left[(x-\xi)^2+\rho^2\right]^{-1/2} d\xi \\
&= -\int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma(\xi)\, \operatorname{sgn}(x-\xi) \frac{\partial}{\partial \xi} \log\left\{|x-\xi| + \left[(x-\xi)^2+\rho^2\right]^{1/2}\right\} d\xi
\end{aligned}
\tag{19}
$$

where $\rho^2 = y^2 + (z+b)^2$. Integrating by parts and assuming $\sigma(-\frac{1}{2}) = \sigma(\frac{1}{2}) = 0$, we obtain, as in Newman [8],

$$
\begin{aligned}
I_{11}(x,y,z) = &\int_{-\frac{1}{2}}^{\frac{1}{2}} \log\left\{|x-\xi| + \left[(x-\xi)^2+\rho^2\right]^{1/2}\right\} \operatorname{sgn}(x-\xi) \frac{\partial}{\partial \xi} \sigma(\xi)\, d\xi \\
&- 2\sigma(x) \log \rho
\end{aligned}
\tag{20}
$$

Then, with the body shape defined by

$$
F(x,\rho) = \epsilon R(x) - \rho = 0 \tag{21}
$$

where the slenderness parameter ϵ is defined as the ratio of the maximum radius to the body length, the limit of (20) for $\rho = O(\epsilon)$ is

$$I_{11}(x,y,z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \log\left[2|x-\xi|\right] \operatorname{sgn}(x-\xi) \frac{\partial}{\partial \xi}\sigma(\xi)\,d\xi$$

$$-2\sigma(x)\log\rho + O(\epsilon^2\sigma) \tag{22}$$

On the body surface we must satisfy the boundary condition (13), which can be written

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial \nu} + O(\epsilon^2\phi) = \epsilon U \frac{\partial R}{\partial x} \tag{23}$$

where $\frac{\partial}{\partial \nu}$ denotes the derivative normal to the curve $r = R(x)$ in the plane $x = \text{constant}$. The normal derivative of (22) is

$$\frac{\partial I_{11}}{\partial \nu} = \frac{\partial I_{11}}{\partial \rho} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2(x-\xi)} \frac{\partial}{\partial \xi}\sigma(\xi)\,d\xi$$

$$-\frac{2\sigma(x)}{\rho} + O(\epsilon^2\sigma) \tag{24}$$

Since $\rho = O(\epsilon)$, the second term is $O(\sigma/\epsilon)$, while the first is $O(\sigma)$. Thus the source strength is found from (23) and (24) to be

$$\sigma(x) = -\frac{\epsilon^2 U}{4\pi}\frac{\partial}{\partial x}S(x) + O(\epsilon^4) \tag{25}$$

where $S(x) = \pi R^2(x)$ is the nondimensional sectional area of the body, and the body boundary condition is satisfied to $O(\epsilon^4)$. We see from (25) that there are certain restrictions on the geometry of the body near the ends if the assumption of $\sigma(-\frac{1}{2}) = \sigma(\frac{1}{2}) = 0$ is to be correct; namely, that the derivative of $S(x)$ must disappear at the ends. If ξ is the distance from the end of the body and

$$r(\xi) \propto \xi^{\alpha}$$

for small ξ ,

then

$$S' \propto r r' \propto \xi^{2\alpha-1}$$

and thus we must have $\alpha > \frac{1}{2}$ for $S' = 0$ at the end. It will be shown below that we must have $S = 0$ at the ends of the body. Tuck [7] presents a detailed discussion of this point.

The boundary conditions (14) and (17) on the bottom and the free surface are not satisfied. By using the source-like Green's function which satisfies these conditions instead of the free-space Green's function, these conditions will be met, but only at the expense of introducing an additional error in the body boundary condition. The required Green's functions satisfy

$$\nabla_x^2 G(x, y, z; \xi, \eta, \zeta) = -4\pi\,\delta(x-\xi)\,\delta(y-\eta)\,\delta(z-\zeta)$$

$$\frac{\partial G}{\partial z} + \frac{U^2}{g}\frac{\partial^2 G}{\partial x^2} = 0 \qquad \text{on} \quad z = 0 \qquad (26)$$

$$\frac{\partial G}{\partial z} = 0 \qquad \text{on} \quad z = -h$$

$$\text{or} \qquad \lim_{z \to -\infty} \nabla_x G = 0$$

as appropriate, and the radiation condition

$$\lim_{x \to +\infty} \nabla_x G = o(x^{-1/2}) \tag{27}$$

The required Green's functions are given by Wehausen and Laitone, [5].

For finite depth

$$G(x,y,z;\xi,\eta,\zeta) = \frac{1}{r} + H(x,y,z;\xi,\eta,\zeta) = \frac{1}{r} + \frac{1}{r_2}$$

$$-\frac{4}{\pi}\int_0^{\pi/2} d\theta \, PV\int_0^{\infty} \frac{e^{-kh}\cosh k(z+h)[\cosh k(\zeta+h)(k\cos^2\theta+\nu)-\nu]}{k\cos^2\theta\cosh kh - \nu\sinh kh}$$

$$\cdot \cos[k(x-\xi)\cos\theta]\cos[k(y-\eta)\sin\theta]\,dk \tag{28}$$

$$-4\int_{\theta_0}^{\pi/2} \frac{e^{-k_0h}\operatorname{sech} k_0h\cosh k_0(z+h)[\cosh k_0(\zeta+h)(k_0\cos^2\theta+\nu)-\nu]}{\cos^2\theta - \nu h\operatorname{sech}^2 k_0h}$$

$$\cdot \sin[k_0(x-\xi)\cos\theta]\cos[k_0(y-\eta)\sin\theta]\,d\theta$$

where PV denotes the principle value,

$$r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$$

$$r_2^2 = (x-\xi) + (y-\eta)^2 + (z+2h+\zeta)^2$$

$$\nu = g/U^2,$$

and k_0 is the real positive root of

$$k_0 - \nu\sec^2\theta\tanh k_0h = 0 \qquad \theta_0 < \theta < \pi/2 \tag{29}$$

where $\theta_0 = \arccos\sqrt{\nu h}$ IF $\nu h = gh/U^2 \leq 1$

$\theta_0 = 0$ IF $\nu h > 1$

In the case of water of infinite depth, the Green's function is

$$G(x,y,z;\xi,\eta,\zeta) = \frac{1}{r} + H(x,y,z;\xi,\eta,\zeta)$$

$$= \frac{1}{r} - \frac{1}{r_1} - \frac{4\nu}{\pi}\int_0^{\pi/2} d\theta \, PV\int_0^{\infty} \frac{e^{k(z+\zeta)}\cos[k(x-\xi)\cos\theta]\cos[k(y-\eta)\sin\theta]}{k\cos^2\theta - \nu}\, dk$$

$$- 4\nu\int_0^{\pi/2} e^{\nu(z+\zeta)\sec^2\theta} \sin[\nu(x-\xi)\sec\theta]\cos[\nu(y-\eta)\sin\theta\sec^2\theta]\sec^2\theta\, d\theta \tag{30}$$

where

$$r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$$

$$r_1^2 = (x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2$$

The functions H are the part of the Green's function which are analytic in the fluid domain and which must be added to the free-space Green's function $1/r$ to satisfy the boundary conditions on the free surface and the bottom.

Using the Green's function above, the potential can be written

$$\phi(x,y,z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma(\xi)\, G(x,y,z;\xi,0,-b)\, d\xi$$

$$= I_{11} + I_{12}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma(\xi) \left[(x-\xi)^2 + \rho^2 \right]^{-1/2} d\xi$$

$$+ \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma(\xi)\, H(x, y, z; \xi, 0, -b) d\xi \qquad (31)$$

The error in meeting the boundary condition on the body is now

$$\frac{\partial I_{12}}{\partial \nu} + O(\epsilon^2 I_{12}) = O(\epsilon^2) \qquad (32)$$

This error is of lower order than the error in the free surface condition, which is $O(\phi^2) = O(\epsilon^4)$. It will be shown in section 5 that this error leads to an error in the pitching moment which is of the same order as the leading term in the moment obtained using the potential (19). The error in the heave force is of higher order.

To obtain the next term in the potential, we consider a line distribution of vertical dipoles of arbitrary strength $\mu(x)$ on the body axis over the interval $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

$$I_2(x, y, z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mu(\xi) \frac{\partial}{\partial z} \left[(x-\xi)^2 + \rho^2 \right]^{-1/2} d\xi \qquad (33)$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial}{\partial \xi} \mu(\xi)\, sgn(x-\xi) \frac{\partial}{\partial z} \log\left\{ |x-\xi| + \left[(x-\xi)^2 + \rho^2 \right]^{1/2} \right\} d\xi$$

$$- 2\mu(x) \frac{\sin\theta}{\rho}$$

by the procedure used above, for $\mu(-\frac{1}{2}) = \mu(\frac{1}{2}) = 0$, where $\theta = \tan^{-1} z+b/y$. In terms of I_{12} and I_2 , the condition (13) becomes

$$\frac{\partial I_{12}}{\partial n} + \frac{\partial I_2}{\partial n} = 0 \quad \text{on} \quad \rho = \epsilon R(x) \tag{34}$$

since (23) is satisfied using I_{11} as ϕ . Expanding I_{12} in a Taylor series in y and z about the body axis and taking account of transverse symmetry, we obtain

$$\frac{\partial I_{12}}{\partial n} = \frac{\partial I_{12}}{\partial \nu} + O(\epsilon^2 I_{12}) = \frac{\partial I_{12}}{\partial z} \sin\theta + O(\epsilon^2 I_{12}) \tag{35}$$

Now

$$\frac{\partial I_2}{\partial n} = \frac{\partial I_2}{\partial \nu} + O(\epsilon^2 I_2) = \frac{\partial I_2}{\partial \rho} + O(\epsilon^2 \mu) \tag{36}$$

$$= 2\mu(x) \frac{\sin\theta}{\rho^2} + O(\epsilon^2 \mu)$$

Thus

$$\mu(x) = -\frac{\epsilon^2 S(x)}{2\pi} \frac{\partial}{\partial z} I_{12}(x, 0, -b) + O(\epsilon^6) \tag{37}$$

It is seen that we have assumed $S(\frac{1}{2}) = S(-\frac{1}{2}) = 0$ in deriving this result. Since from (35) the boundary condition on the body is satisfied to order $O(\mu/\epsilon^2) = O(\epsilon^2)$, there is no inconsistency in including a term $O(\epsilon^4)$ in the potential. Since the error in the free surface is also $O(\epsilon^4)$, we may use just the free-space dipole Green's function.

The error in force and moment resulting from neglecting the free surface condition for the dipole distribution will be shown to be of higher order.

In the usual slender-body approach (Tuck, [7]), it is necessary to find an inner expansion for the potential and match the outer limit of this solution with the inner limit of the outer solution, which is the line singularity distribution. We have been able to avoid this because of the particularly simple form of the boundary condition to be met on a body of revolution, and hence have been able to use the inner limit of the outer solution to satisfy the boundary condition directly.

4. APPROXIMATE FAR-FIELD SOLUTION USING GREEN'S THEOREM

The results obtained in the preceding section can be obtained, in the far-field, by applying Green's theorem. This more direct approach, in which it is not necessary to assume a form of the solution a priori, is of considerable interest. It is necessary only to apply the slender-body assumption in the approximate evaluation of the surface integral in Green's theorem, which is an exact result. Green's theorem has been used to find the near-field potential in various slender-body problems involving a free surface by Vossers [8], Newman [6], [9], and Joosen [10].

If S is a surface enclosing a volume V in which the disturbance potential satisfies Laplace's equation (11), then the potential inside V is given in terms of the values of the potential and its normal derivative on the surface S by

$$\phi(x,y,z) = -\frac{1}{4\pi}\iint_S \Big[G(x,y,z;\xi,\eta,\zeta)\frac{\partial\phi}{\partial n}(\xi,\eta,\zeta) - \phi(\xi,\eta,\zeta)\frac{\partial}{\partial n} G(x,y,z;\xi,\eta,\zeta)\Big] dS \qquad (38)$$

where $\partial/\partial n$ denotes the inward normal derivative, and G is the Green's function given in the preceding section. We take as the surface S the body surface S_S, the free surface S_{FS}, the bottom S_B, and a vertical cylindrical surface S_R of radius R centered at the origin surrounding the body, and a tube connecting S_S and S_{FS} which we shrink down to a vanishing line. The surface is shown in Figure 2.

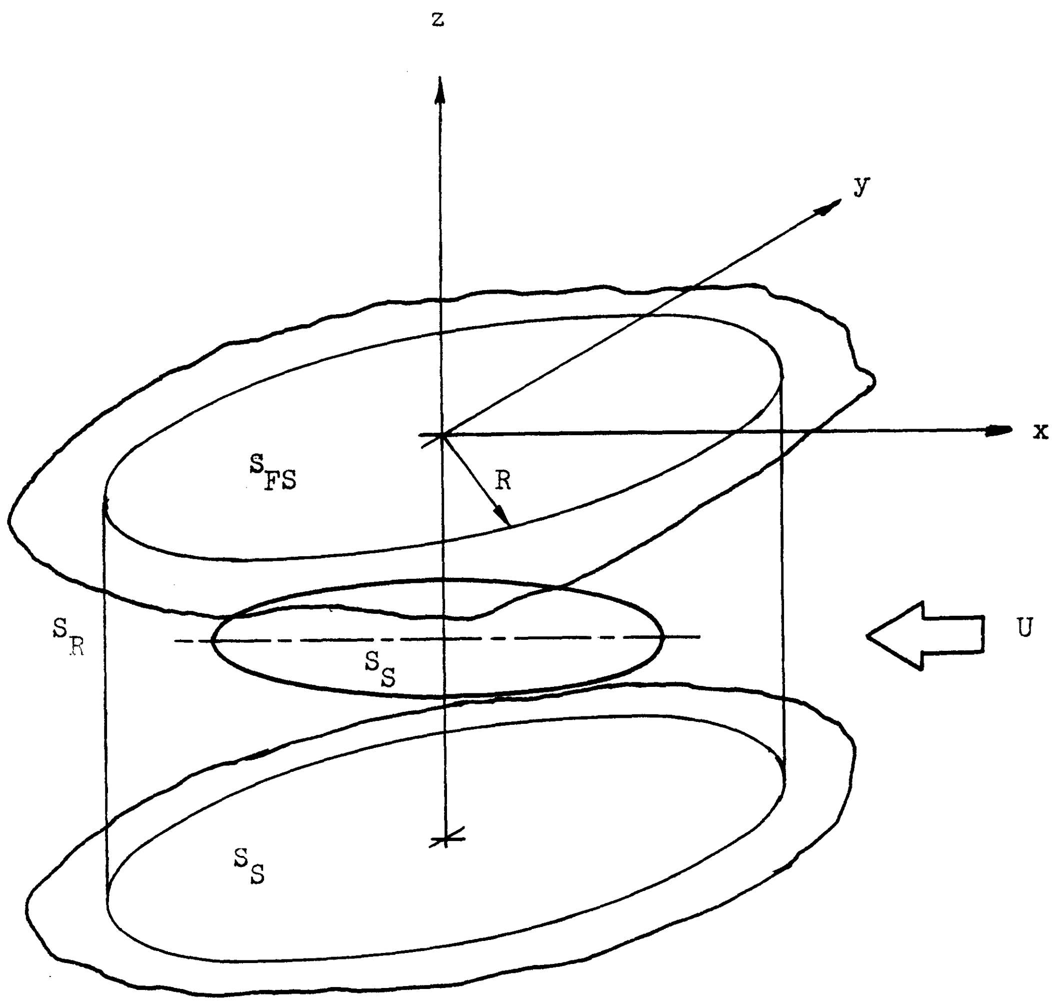


Figure 2 The Surface S for Applying Green's Theorem

The integral over S_B is zero due to the bottom boundary condition (14). On the free surface

$$\iint_{S_{FS}} \left(G \frac{\partial \phi}{\partial n} - \phi \frac{\partial G}{\partial n} \right) dS = -\frac{U^2}{g} \oint \left(G \frac{\partial \phi}{\partial \xi} - \phi \frac{\partial G}{\partial \xi} \right) dc \tag{39}$$

where the integral is taken around the intersection of S_{FS} and S_R. In the limit as $R \to \infty$, this integral is zero since both ϕ and G satisfy the radiation condition. For the same reason the integral over the surface S_R also vanishes in the limit $R \to \infty$, and we are left with the integral over the surface of the body. Thus we want to solve the integral equation

$$\begin{aligned} \phi(x, y, z) = -\frac{1}{4\pi} \iint_{S_s} \Big[& G(x, y, z; \xi, \eta, \zeta) \frac{\partial \phi}{\partial n}(\xi, \eta, \zeta) \\ & - \phi(\xi, \eta, \zeta) \frac{\partial G}{\partial n}(x, y, z; \xi, \eta, \zeta) \Big] dS \end{aligned} \tag{40}$$

We will find an approximate solution valid in the far-field for small values of the slenderness parameter ϵ.

We split the integral into two parts

$$\phi(x, y, z) = J_1 + J_2 \tag{41}$$

where

$$J_1 = -\frac{1}{4\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\pi} G \frac{\partial \phi}{\partial n} \, d\theta \, d\xi \tag{42}$$

and

$$J_2 = \frac{1}{4\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\pi} \phi \frac{\partial G}{\partial n} \, d\theta \, d\xi \tag{43}$$

From the body boundary condition (23), we have

$$\frac{\partial \phi}{\partial n} = \epsilon U \frac{\partial R}{\partial x} + O(\epsilon^3) \tag{44}$$

In the integral J_1 we expand $G(x,y,z;\xi,\eta,\zeta)$ in a Taylor series in η and ζ about $\eta = 0$ and $\zeta = -b$ and obtain

$$\begin{aligned} J_1(x,y,z) = -\frac{\epsilon^2 U}{4\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\pi} \Big[& G(x,y,z;\xi,0,-b) \\ & + \epsilon R \cos\theta \frac{\partial}{\partial \eta} G(x,y,z;\xi,0,-b) + \epsilon R \sin\theta \frac{\partial}{\partial \zeta} G(x,y,z;\xi,0,-b) \Big] \\ & \cdot \frac{\partial R(\xi)}{\partial \xi} R(\xi) \, d\theta \, d\xi + O(\epsilon^4) \end{aligned} \tag{45}$$

Since R is constant on a body of revolution for ξ constant, the second and third terms integrate out, and we have the result

$$J_1(x,y,z) = -\frac{\epsilon^2 U}{4\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} G(x,y,z;\xi,0,-b) \frac{\partial}{\partial \xi} S(\xi) \, d\xi + O(\epsilon^4) \tag{46}$$

This potential, a distribution of sources on the axis proportional to the derivatives of the sectional area curve, is the usual result of slender-body theory (Tuck, [7]). Since in the above, we have expanded the $1/r$

term in a Taylor series, these results are valid only for small ϵ/r .

It is sometimes more convenient to have the integral in terms of $S(\xi)$. Thus, integrating by parts, we obtain

$$J_1(x,y,z) = \frac{\epsilon^2 U}{4\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} S(\xi) \frac{\partial}{\partial \xi} G(x,y,z;\xi,0,-b)\, d\xi \tag{47}$$

This represents an axial distribution of doublets (cf. Batchelor, [11]).

We now consider the integral J_2 , equation (43). Using J_1 for ϕ in this integral, we see that

$$J_2 = O(\epsilon J_1) = O(\epsilon^3) \tag{48}$$

and thus we need use only the J_1 term of ϕ to evaluate the lowest order term of J_2 . Since Green's theorem for a point (x,y,z) on the surface S is

$$\phi = \frac{1}{2\pi} \iint_S \left(\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n} \right) dS , \tag{49}$$

we must take

$$\phi = 2 J_1 \tag{50}$$

in the evaluation of equation (43). We divide J_1 into two parts, J_{11} and J_{12} , corresponding to the $1/r$ and H terms of the Green's function. Then we write J_2 as the sum of J_{21} and J_{22} corresponding to I_{11} and I_{12} respectively. Thus

$$J_{21}(x,y,z) = -\frac{\epsilon}{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_0^{2\pi} J_{11}(\xi,\eta,\zeta)\frac{\partial}{\partial\eta}G(x,y,z;\xi,\eta,\zeta)R(\xi)\,d\theta\,d\xi \tag{51}$$

Then

$$\frac{\partial G}{\partial\nu} = \frac{\partial G}{\partial\eta}\cos\theta + \frac{\partial G}{\partial\zeta}\sin\theta \tag{52}$$

Expanding these derivatives in Taylor series about $\eta = 0$, $\zeta = -b$, we obtain

$$J_{21}(x,y,z) = \frac{\epsilon}{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_0^{2\pi} J_{11}(\xi,\eta,\zeta)\Big[\frac{\partial}{\partial\eta}G(x,y,z;\xi,0,-b)\cos\theta$$

$$+\frac{\partial}{\partial\zeta}G(x,y,z;\xi,0,-b)\Big]R(\xi)\,d\theta\,d\xi \quad + O(\epsilon^4) \tag{53}$$

J_{11} is a function of x and ρ only. Hence it is a constant on each section on a body of revolution, and the first two terms in J_{11} integrate to zero leaving $J_{21} = O(\epsilon^4)$.

Finally, we consider

$$J_{22}(x,y,z) = \frac{\epsilon}{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_0^{2\pi} J_{12}(\xi,\eta,\zeta)\frac{\partial}{\partial\eta}G(x,y,z;\xi,\eta,\zeta)R(\xi)\,d\theta\,d\xi \tag{54}$$

Expanding I_{12} and $\partial G/\partial n$ in a Taylor series about the body axis, we have

$$J_{22}(x,y,z) = \frac{\epsilon}{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{0}^{2\pi}\Big[J_{11}(\xi,0,-b)\frac{\partial}{\partial \eta}J_{11}(\xi,0,-b)\,\epsilon R\cos\theta$$

$$+\frac{\partial}{\partial \zeta}J_{11}(\xi,0,-b)\,\epsilon R\sin\theta\Big]\Big[\frac{\partial}{\partial \eta}G(x,y,z;\xi,0,-b)\cos\theta \tag{55}$$

$$+\frac{\partial}{\partial \zeta}G(x,y,z;\xi,0,-b)\sin\theta\Big]R(\xi)\,d\theta\,d\xi \quad + O(\epsilon^4)$$

Then integrating and noting $\frac{\partial}{\partial \eta} I_{12} = 0$ due to transverse symmetry of the H-term of the Green's function, we obtain the result

$$J_{22} = \frac{\epsilon^2}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\partial}{\partial \zeta}J_{12}(\xi,0,-b) \tag{56}$$

$$+\frac{\partial}{\partial \zeta}G(x,y,z;\xi,0,-b)\,R^2(\xi)\,d\xi + O(\epsilon^4)$$

Since $J_{12} = O(\epsilon^2)$, we have $J_{22} = O(\epsilon^4)$. Because we have been discarding terms $O(\epsilon^4)$ all along, to be consistent we should drop J_{22} also. However, it will be shown in the next section that J_{22} leads to a contribution to the moment of leading order, while the discarded terms will lead to higher order force and moment terms. The part of J_{22} due to the H-term of the Green's function in equation (42) also leads to a higher order force and moment and hence need not be included. In addition, in the previous section it was shown that in the near-field the dipole term is needed to satisfy the boundary condition on the body to the same order as the free surface condition.

5. FORCE AND MOMENT

The force and moment on the body may be calculated by several methods. We shall simply integrate the pressure over the body. The vertical force and pitching moment are given by

$$F_z = -\iint_{S_B} p\, n_z\, dS$$
$$M_y = -\iint_{S_B} p\, (\underline{r} \times \underline{n})_y\, dS \tag{57}$$

where $\underline{n}$ is the outward unit normal on the body.

For a slender body of revolution whose shape is defined by (21),

$$n_x = O(\epsilon)$$
$$n_y = \cos\theta + O(\epsilon)$$
$$n_z = \sin\theta + O(\epsilon)$$

where θ is defined as above. Thus the integral can be written

$$F_z = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\pi} p \sin\theta \; r\, d\theta\, dx \; + O(\epsilon^2 p)$$
$$= -\int_{-\frac{1}{2}}^{\frac{1}{2}} f_z(x)\, dx + O(\epsilon^2 p) \tag{58}$$
$$M_y = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\pi} x p \sin\theta \; r\, d\theta\, dx \; + O(\epsilon^2 p)$$
$$= -\int_{-\frac{1}{2}}^{\frac{1}{2}} x f_z(x)\, dx + O(\epsilon^2 p)$$

since $r = O(\epsilon)$ on the body. From Bernoulli's equation the pressure is

$$p = -\frac{\rho}{2} \left|\nabla \Phi\right|^2 - \rho g z \tag{59}$$

The ρgz term yields the hydrostatic force and moment

$$\begin{aligned} \rho g \forall &= -\rho g \iint_{S_B} z\, n_z\, dS \\ \rho g \forall x_0 &= -\rho g \iint_{S_B} z\, (\underline{r} \times \underline{n})_y\, dS \end{aligned} \tag{60}$$

where $\forall$ is the body volume, and x_o is the x-coordinate of the center of buoyancy. We will neglect this hydrostatic force and moment in the following.

In terms of the disturbance potential ϕ , the pressure is

$$p = -\frac{\rho}{2}\left[U^2 - 2U\phi_x + \phi_x^2 + \phi_y^2 + \phi_z^2\right] \tag{61}$$

The U^2 term is a constant and hence is integrated out. The terms ϕ_x^2, ϕ_y^2, and ϕ_z^2 are higher order and will be neglected. Thus the dynamic pressure to order is

$$p = \rho U \phi_x \tag{62}$$

The disturbance potential near the body has been found to be

$$\begin{aligned} \phi(x,y,z) = {} & -2\sigma(x)\log r + \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma(\xi)\, H(x,0,-b;\xi,0,-b)\, d\xi \\ & + \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma(\xi) \frac{\partial}{\partial z} H(x,0,-b;\xi,0,-b)\, r(\xi) \sin\theta\, d\xi - 2\mu(x)\frac{\sin\theta}{r} + O(\epsilon^4) \end{aligned} \tag{63}$$

where now $r^2 = y^2 + (z+b)^2$ and σ and μ are given by (25) and (37) respectively. Then

$$\phi_x(x,y,z) = -2\frac{\partial\sigma(x)}{\partial x}\log r + \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma(\xi)\frac{\partial}{\partial x}H(x,0,-b;\xi,0,-b)\,d\xi$$

$$+\int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma(\xi)\frac{\partial}{\partial x}\frac{\partial}{\partial z}H(x,0,-b;\xi,0,-b)\,r(\xi)\sin\theta\,d\xi \qquad (64)$$

$$-2\frac{\partial}{\partial x}\mu(x)\frac{\sin\theta}{r}$$

On a body of revolution, the first and second terms of (64) are functions of x only and hence integrate out. The third term yields

$$-\rho U\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma(\xi)\frac{\partial}{\partial x}\frac{\partial}{\partial z}H(x,0,-b;\xi,0,-b)\,r^2(x)\sin^2\theta\,d\xi\,d\theta\,dx$$

(65)

$$= -\frac{\epsilon^4\rho U^2}{4\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} S(x)\int_{-\frac{1}{2}}^{\frac{1}{2}} S(\xi)\frac{\partial}{\partial x}\frac{\partial}{\partial z}\frac{\partial}{\partial \xi}H(x,0,-b;\xi,0,-b)\,d\xi\,dx$$

The contribution to the moment from this term is

$$-\rho U\int_{-\frac{1}{2}}^{\frac{1}{2}} x\int_0^{2\pi}\int_0^{\frac{1}{2}} \sigma(\xi)\frac{\partial}{\partial x}\frac{\partial}{\partial z}H(x,0,-b;\xi,0,-b)\,r^2(x)\sin^2\theta\,d\xi\,d\theta\,dx$$

(66)

$$= -\frac{\epsilon^4\rho U^2}{4\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} x\,S(x)\int_{-\frac{1}{2}}^{\frac{1}{2}} S(\xi)\frac{\partial}{\partial x}\frac{\partial}{\partial z}\frac{\partial}{\partial \xi}H(x,0,-b;\xi,0,-b)\,d\xi\,dx$$

From the last term of ϕ_x , we have the force contribution

$$2\rho U \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial}{\partial x} \mu(x) \int_0^{2\pi} \sin^2\theta \, d\theta \, dx$$

$$= 2\pi\rho U \left[\mu\left(-\tfrac{1}{2}\right) - \mu\left(\tfrac{1}{2}\right) \right] = 0 \qquad (67)$$

according to the assumptions made in the derivation in section 3. The corresponding contribution to the moment is

$$2\rho U \int_{-\frac{1}{2}}^{\frac{1}{2}} x \frac{\partial}{\partial x} \mu(x) \int_0^{2\pi} \sin^2\theta \, d\theta dx$$

$$= -2\pi\rho U \int_{-\frac{1}{2}}^{\frac{1}{2}} \mu(x) \, dx \qquad (68)$$

$$= \frac{\epsilon^2 \rho U^2}{4\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} S(x) \int_{-\frac{1}{2}}^{\frac{1}{2}} S(\xi) \frac{\partial}{\partial z} \frac{\partial}{\partial \xi} H(x,0,-b;\xi,0,-b) d\xi dx$$

These expressions are all $O(\epsilon^4)$. The ϕ_x^2, ϕ_y^2, and ϕ_z^2 terms in (61) lead to force and moment terms $O(\epsilon^5)$ and higher. The free surface terms resulting from the dipole distribution are $O(\epsilon^4)$ and clearly will result in higher order contributions to the force and moment and hence can be neglected. Combining (66) and (68), we have the result

$$M = \frac{\epsilon^4 \rho U}{4\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} S(x) \int_{-\frac{1}{2}}^{\frac{1}{2}} S(\xi) \left[\frac{\partial}{\partial z} \frac{\partial}{\partial \xi} H(x,0,-b;\xi,0,-b) \right.$$

$$\left. - x \frac{\partial}{\partial x} \frac{\partial}{\partial z} \frac{\partial}{\partial \xi} H(x,0,-b;\xi,0,-b) \right] d\xi \, dx \qquad (69)$$

It is of some interest to obtain these results by applying Lagally's theorem (Cummins, [12]). The force on a source is equal to $-4\pi\rho \underline{c} \boldsymbol{\sigma}$, where $\underline{c}$ is the flow velocity due to external singularities, and σ is the source strength. To lowest order, the force on a dipole is zero, and the moment is given by $-4\pi\rho c\mu$, where now c is the external flow velocity component normal to the dipole axis, in this case the stream velocity U, to lowest order, and μ is the dipole strength. The "Lagally" force and moment on the source distribution are easily found to be, to lowest order,

$$F_{SL} = -\frac{\epsilon^4 \rho U^2}{4\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} S(x)\int_{-\frac{1}{2}}^{\frac{1}{2}} S(\xi)\frac{\partial}{\partial x}\frac{\partial}{\partial z}\frac{\partial}{\partial \xi} H(x,0,-b;\xi,0,-b)\,d\xi\,dx$$

$$M_{SL} = -\frac{\epsilon^4 \rho U^2}{4\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} S(x)\int_{-\frac{1}{2}}^{\frac{1}{2}} S(\xi)\left[x\frac{\partial}{\partial x}\frac{\partial}{\partial z}\frac{\partial}{\partial \xi} H + \frac{\partial}{\partial z}\frac{\partial}{\partial \xi} H\right] d\xi\,dx \tag{70}$$

Similarly, the force and moment due to the dipole distribution are

$$F_{DL} = 0$$

$$M_{DL} = \frac{\epsilon^4 \rho U^2}{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} S(x)\int_{-\frac{1}{2}}^{\frac{1}{2}} S(\xi)\frac{\partial}{\partial x}\frac{\partial}{\partial \xi} H(x,0,-b;\xi,0,-b)\,d\xi\,dz$$

The expressions for the force are the same as obtained by integrating the linearized pressure, (65) and (67), but the moments due to the source and dipole distributions are not those obtained previously. However, the sum

$$M_L = M_{SL} + M_{DL}$$

is that given by (69) and thus the final results are identical. This discrepancy in the intermediate results for the moment is a direct consequence of the fact that the source distribution (19) alone does not satisfy the body boundary condition. In the first calculation we found the force and moment on the surface which was to be represented by the source distribution, while the application of Lagally's theorem has given the force and moment acting on the surface which is actually represented by the source distribution, which clearly is significantly different from what is wanted. Similarly, the integral of the pressure over the intended body surface gives the contribution to the moment due to the dipoles, while the Lagally moment on the dipoles does not correspond to the moment on any given body, since a line distribution of vertical dipoles in a uniform horizontal stream by itself does not by itself represent a body, but rather represents the effect of a correction to the shape of the body represented by the source distribution.

6. NUMERICAL RESULTS

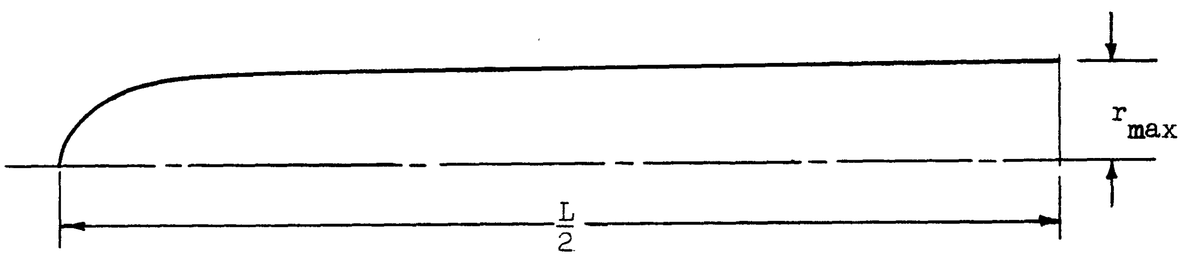
A program has been written to evaluate the expressions for the force and moment on a body of revolution moving horizontally in water of finite depth, equations (65) and (69). The program is listed in the appendix.

Figures 4, 5, 6, and 7 show the force and moment on a Rankine ovoid of length-diameter ratio 10.5 for two different submergences in infinitely deep water, along with the experimental results obtained by Korvin-Kroukovsky, et al., [13] for comparison. The shape of the body is shown in Figure 3. The agreement between calculated and measured results for the moment is very good. The agreement for the force is not as good. Figures 5 and 7 also show the results of computations by Pond [2] who used the exact uniform flow potential (a source and a sink) instead of the source distribution obtained by using the slender-body approximation. The dipole terms were calculated in the same manner as was used above, based on a heuristic argument. Both the moment on the source-sink combination alone, calculated by Lagally's theorem (labeled "Pond's First Result"), and the moment on the source, sink, and dipoles are given. It is clear from these curves that the source potential alone is insufficient from a practical point of view. The calculations using equation (69) agree very closely with Pond's results in spite of the difference in the source potential and the bluntness of the body.

Figures 8 and 9 and Table 1 present the results of calculations for Froude number 0.55, depth of submergence varying from 0.144L to 2.5L, and depth of water varying from 0.5L to infinite depth. It is clear that for a body near the surface the effect of the bottom is not felt if the water is deeper than about two body lengths. Furthermore, from Table 1 it is clear that if the body submergence is more than about one or two body lengths the force and moment due to the free surface are very small, unless the body is close to the bottom, in which case the force and moment are due to the wall effect of the bottom.

Surface of water corresponding to submergence b = 0.266L

Surface of water corresponding to submergence b = 0.144L



Only one quarter of body is shown

length/diameter = 10.5

Figure 3 Shape of Rankine Ovoid used in Calculations

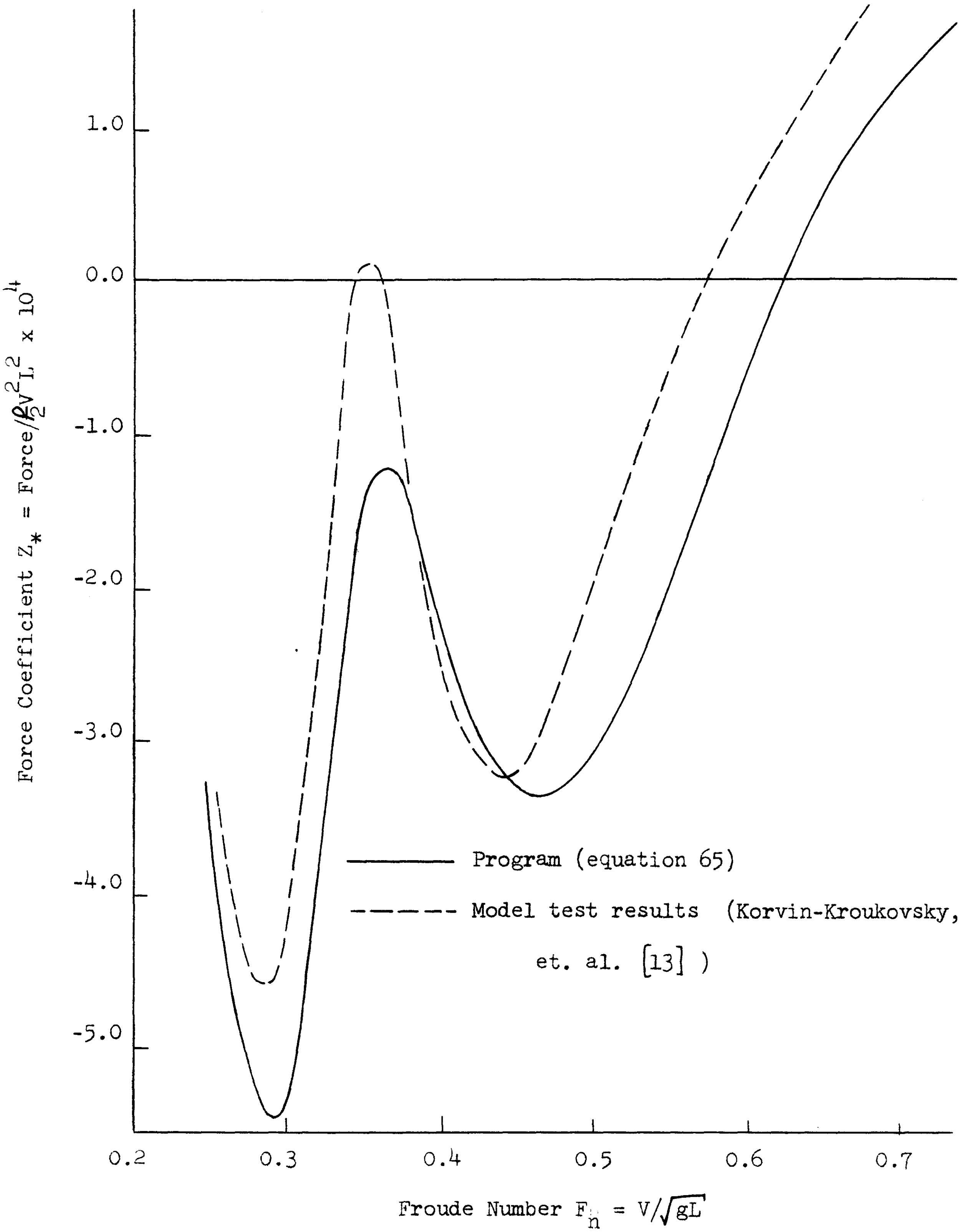


Figure 4 Heave Force on Rankine Ovoid at 0.144L Submergence

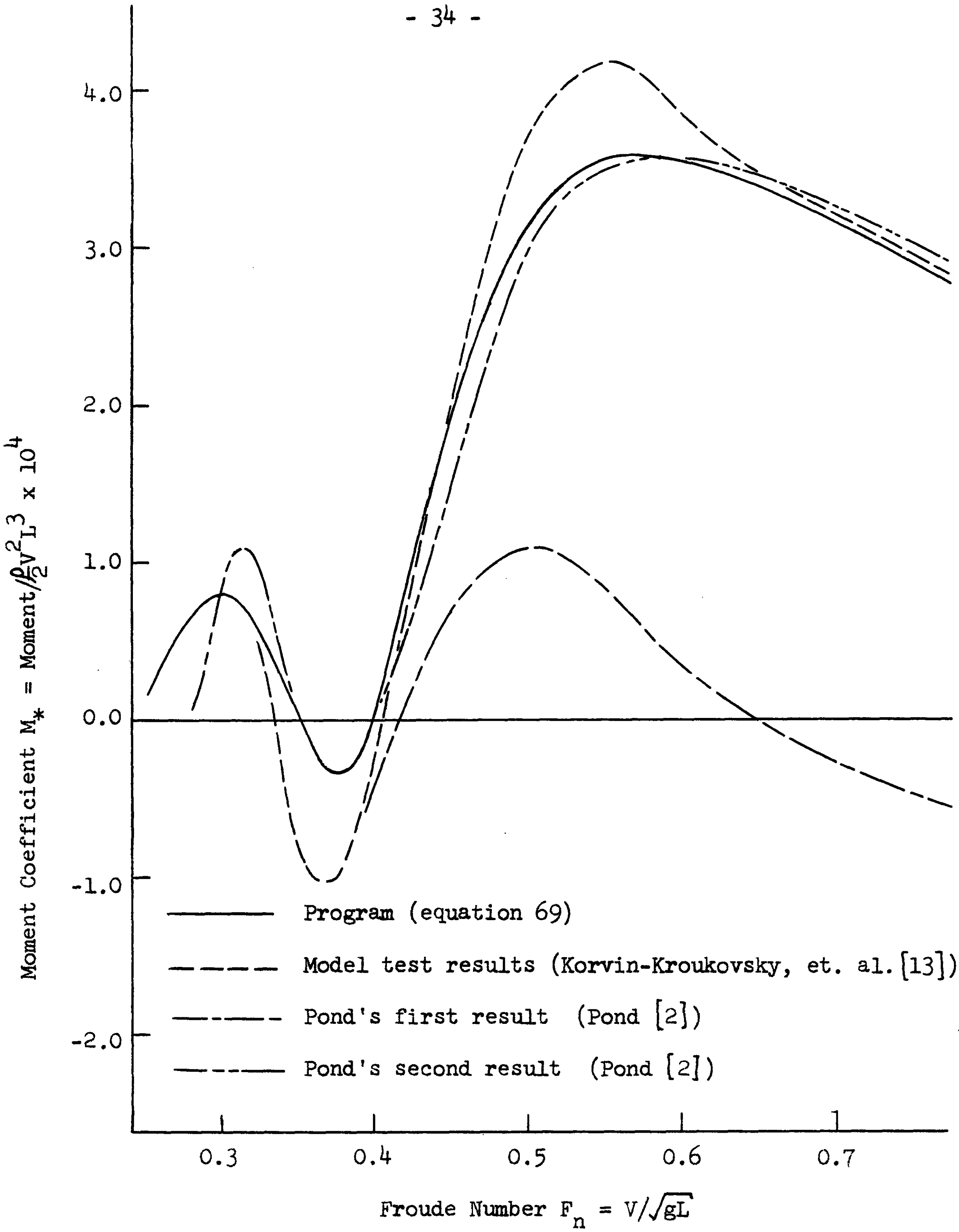


Figure 5 Pitching Moment on Rankine Ovoid at 0.144L Submergence

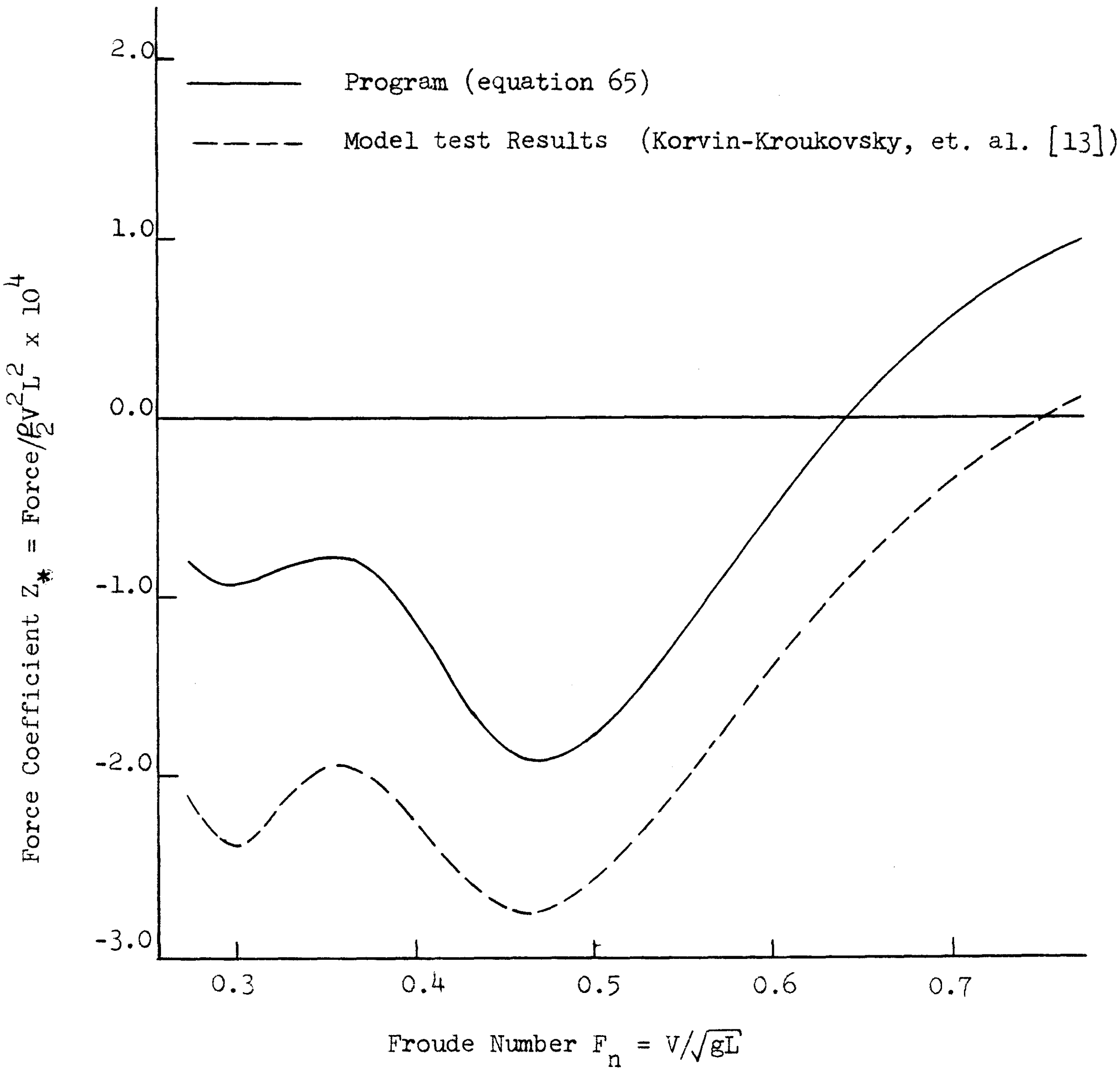


Figure 6 Heave Force on Rankine Ovoid at 0.266L Submergence

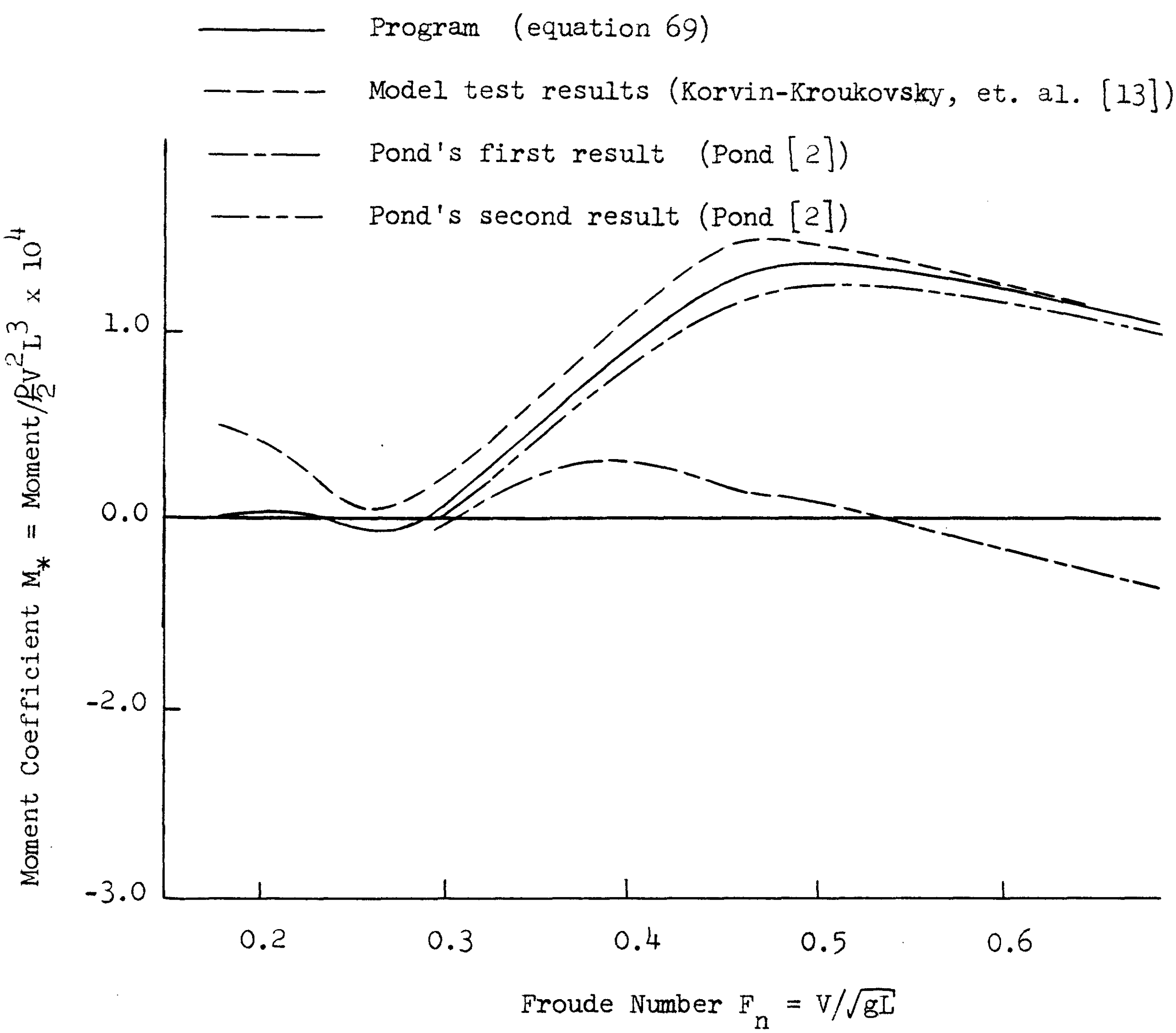


Figure 7 Pitching Moment on Rankine Ovoid at 0.266L Submergence

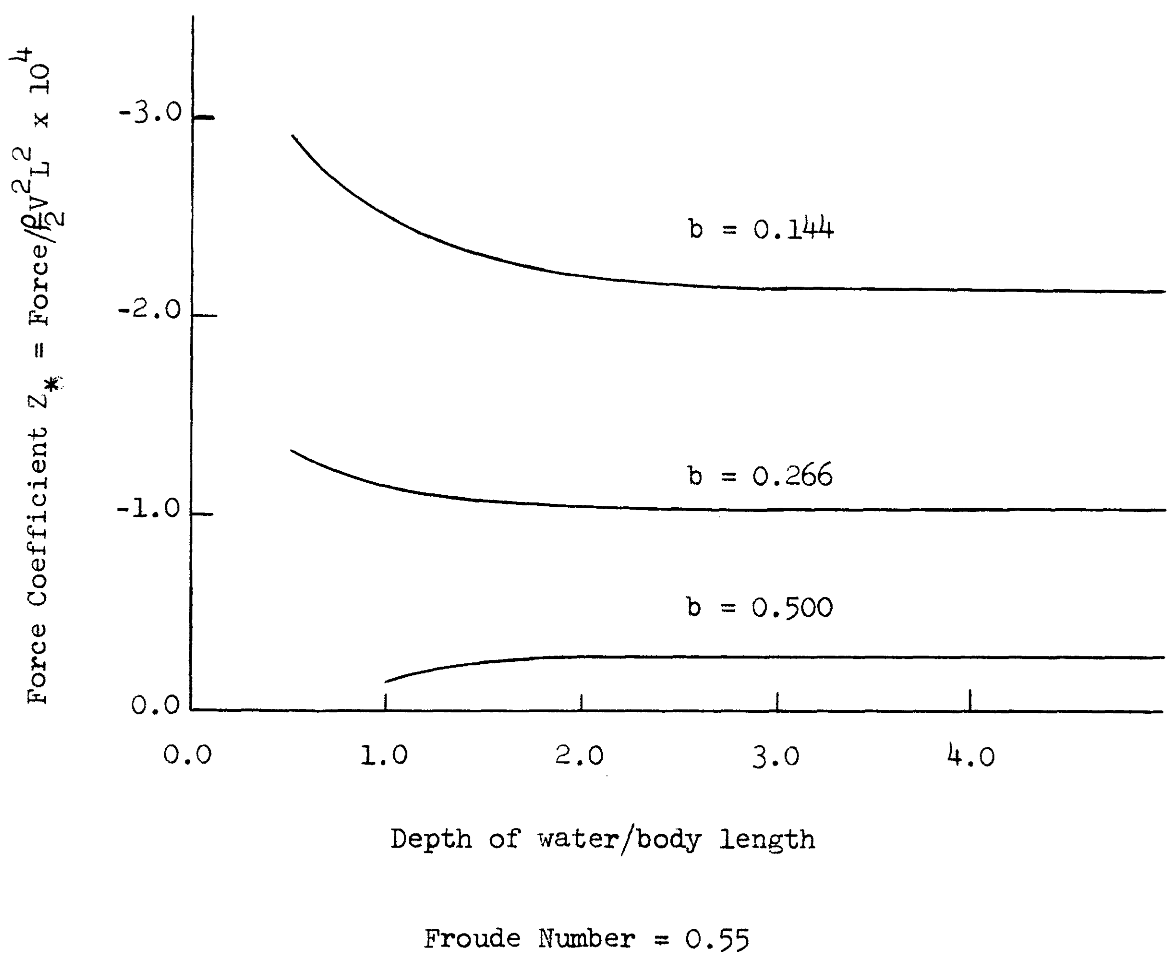


Figure 8 Heave Force on Rankine Ovoid as a Function of Submergence and Depth of Fluid

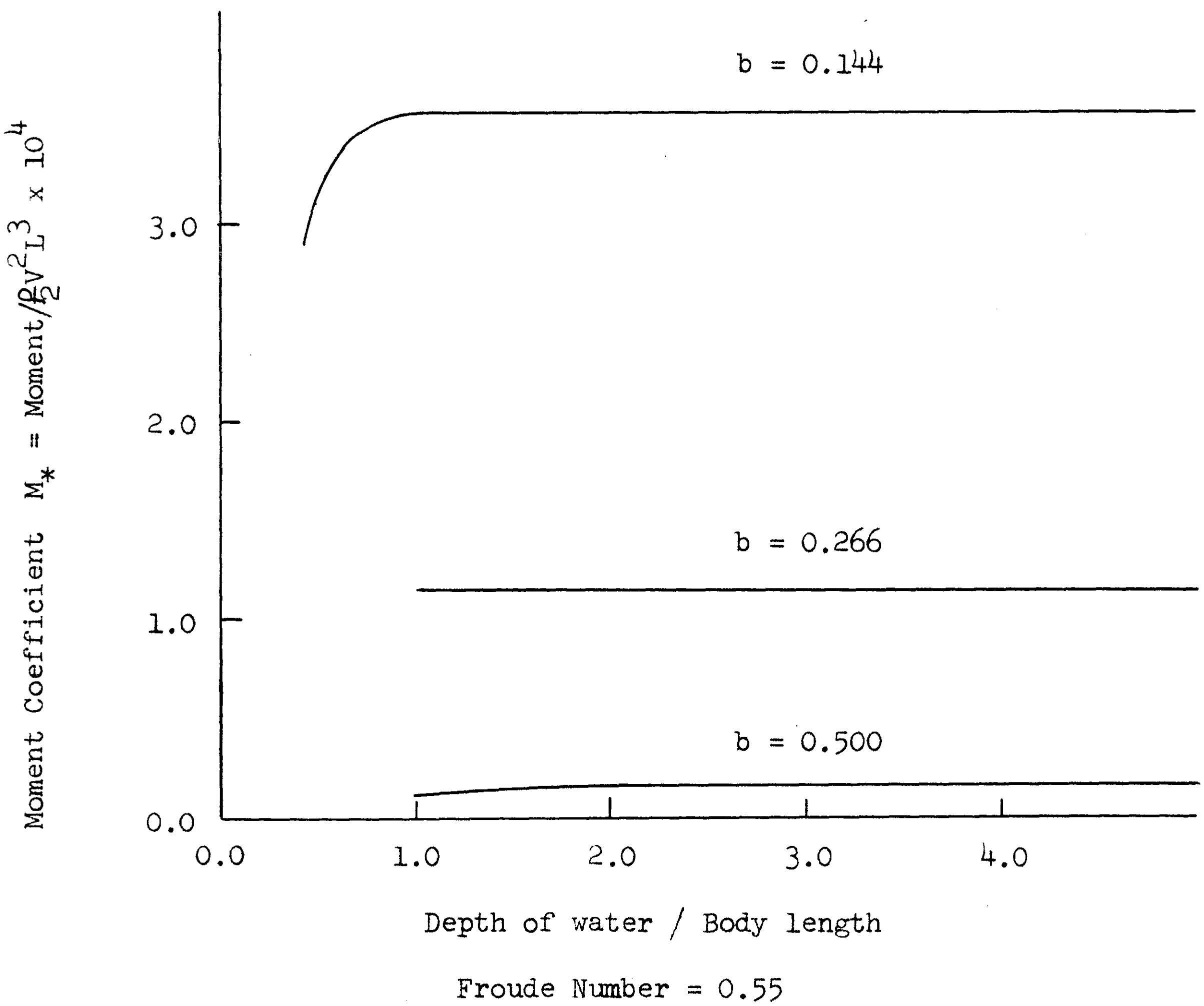


Figure 9 Pitching Moment on Rankine Ovoid as a function of Submergence and Depth of Water

TABLE 1.

EFFECT OF SUBMERGENCE AND DEPTH OF WATER
ON FORCE AND MOMENT ON RANKINE OVOID

Froude Number = 0.55

Submergence	Depth of Water	Force Coefficient	Moment Coefficient
1.0L	2.0L	-1.22×10^{-6}	5.11×10^{-8}
	3.0L	-2.59×10^{-6}	3.86×10^{-7}
	Deep Water	-2.86×10^{-6}	4.10×10^{-7}
1.5L	2.0	9.48×10^{-6}	-3.97×10^{-6}
	3.0	-1.29×10^{-7}	-6.24×10^{-8}
	Deep Water	-4.67×10^{-7}	1.20×10^{-8}
2.0L	3.0	9.54×10^{-7}	-3.47×10^{-7}
	Deep Water	-1.26×10^{-7}	3.76×10^{-10}
2.5L	3.0	9.65×10^{-6}	-3.98×10^{-6}
	Deep Water	-4.72×10^{-8}	1.22×10^{-11}

7. CONCLUSION

The theory of Havelock and Pond for the force and moment on a body of revolution moving under a free surface has been shown to be a consistent first-order theory, while theories derived ignoring the effect of the body generated waves on the flow near the body are not. The theory has been extended to the case of an arbitrary slender body of revolution and to include the effect of finite depth of water. This more general theory compares very well with the theory of Pond and with experimental results for the special case of a Rankine ovoid in infinite depth water.

8. REFERENCES

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APPENDIX

PROGRAM LISTING

```
C
C *** PROGRAM TO CALCULATE HEAVE FORCE AND PITCH MOMENT ON SUBMERGED
C     SLENDER BODY OF REVOLUTION MOVING IN WATER OF FINITE DEPTH.
C
C *** FINAL VERSION
C
      DIMENSION XBB(101),SMRS(101),RAD(101),H2(210),H3(210)
      DIMENSION LABEL(20)
      COMMON/CITG/H,B,Y,YPH,BPH,XMA,GNU,GNUH
      PI = 3.14159
C
C *** INPUT
C
    1 CONTINUE
      READ(5,510) (LABEL(I),I = 1,20)
      WRITE(6,510) (LABEL(I),I = 1,20)
      READ (5,520) NORD,EL
      WRITE(6,520) NORD,EL
      DO 360 I=1,NORD
  360 READ(5,540) RAD(I)
      WRITE(6,530) (RAD(I),I = 1,NORD)
      DO 330 I = 1,NORD
  330  RAD(I) = RAD(I)/EL
      WRITE(6,530) (RAD(I),I = 1,NORD)
    2 CONTINUE
      READ (5,550)  FN,H,B,NCHECK
      WRITE(6,550)  FN,H,B,NCHECK
      H=H/EL
      B=B/EL
      Y = B
C
C *** CALCULATE ORDINATES
C
      NSP = NORD-1
      FNSP = FLOAT(NSP)
      DO 340 I = 1,NORD
```

```
  340 XBB(I)=-0.5 + (FLOAT(I)-1.0)/FNSP
C
C *** CALCULATE PRODUCT OF SIMPSON'S MULTIPLIERS AND RADIUS SQUARED.
C
      DO 350 I = 1,NORD
      NTEST=I-(I/2)*2
      SWT = 4.0
      IF(NTEST.EQ.1)  SWT=2.0
      IF(I.EQ.1.OR.I.EQ.NORD) SWT = 1.0
      WRITE(6,530) SWT
  350 SMRS(I) = RAD(I)*RAD(I)*SWT
C
C *** CALCULATE NU
C
      GNU = 1.0/FN/FN
      GNUH = GNU*H
C
C *** FIND LOWER LIMIT OF INTEGRATION
C
      IF(GNUH-1.0)  300,310,310
  300 E = ARCOS(SQRT(GNUH))
      GO TO 320
  310 E = 0.0
  320 CONTINUE
C
C *** CALCULATE DERIVATIVES OF ANALYTIC PART OF GREEN'S FUNCTION
C
      DO 200 I = 1,NORD
      K = I-1
      NRDP=NORD+K
      NRDM=NORD-K
      XMA = FLOAT(K)/FNSP
      CALL  HDER(H2(NRDP),H2(NRDM),H3(NRDP),H3(NRDM),E)
      WRITE(6,201)  H2(NRDP),H2(NRDM),H3(NRDP),H3(NRDM)
  201 FORMAT(4E15.7)
  200 CONTINUE
```

```
C
C *** OUTER SUMMATION
C
      SOF = 0.0
      SOM = 0.0
      DO 110 I = 1,NORD
C
C *** INNER SUMMATION
C
      SIF = 0.0
      SIM = 0.0
      DO 100 J = 1,NORD
      N = I-J+NORD
      C1 = SMRS(J)*H3(N)
      SIF =  SIF + C1
  100 SIM = SIM + SMRS(J)*H2(N)-XBB(I)*C1
      SOF =  SOF + SMRS(I)*SIF
  110 SOM =  SOM + SMRS(I)*SIM
C
C *** CALCULATE FINAL RESULTS
C
      CONST = PI/18.0/FNSP/FNSP
      ANSF =  CONST*SOF
      ANSM = CONST*SOM
      WRITE(6,500) (LABEL(I),I = 1,20),FN,H,B,N,ANSF,ANSM
      IF(NCHECK.EQ.1) GO TO 1
      IF(NCHECK.EQ.2) GO TO 2
  500 FORMAT(1H1///1H ,119(1H*)/3H * ,115(1H*),2H *,2(/4H * *,113X,3H* *
     1)/4H * *,16X,20A4,17X,3H* */4H * *,113X,3H* */4H * *,7X,16HFROUDE
     2 NUMBER = ,F10.6,7X,8HDEPTH = ,F10.6,7X,14HSUBMERGENCE = ,F10.6,7X
     3,4HN = ,I5,8X,3H* */4H * *,113X,3H* */4H * *,10X,20HFORCE COEFFICI
     4ENT = ,E15.7,21X,21HMOMENT COEFFICIENT = ,E15.7,11X,3H* *,2(/4H *
     5*,113X,3H* *)/3H * ,115(1H*),2H */1H ,119(1H*)/1H1)
  510 FORMAT(20A4)
  520 FORMAT(I5,F10.6)
  530 FORMAT(8F10.6)
```

```
540 FORMAT(F10.6)                                        MAIN 109
550 FORMAT(3F10.4,I5)                                    MAIN 110
    STOP                                                 MAIN 111
    END                                                  MAIN 112
```

```
      SUBROUTINE HDER(H2P,H2M,H3P,H3M,E)                                HDER
C     FOR PROG B  H3                                                    HDER
      COMMON/CITG/H,B,Y,YPH,BPH,XMA,GNU,GNUH                            HDER
      PDF = 1.2732394                                                   HDER
      YPH = Y+H                                                         HDER
      BPH = B+H                                                         HDER
      XMAS = XMA*XMA                                                    HDER
      RP = Y + 2.0*H + B                                                HDER
      RTS = XMAS + RP*RP                                                HDER
      TERM=  3.0*RP/(RTS**2.5)                                          HDER
      H22 = TERM*XMA                                                    HDER
      H32 = TERM*(1.0-5.0*XMAS/RTS)                                     HDER
      CALL TERM3(H23,H33)                                               HDER
      CALL TERM4(E,H24,H34)                                             HDER
      H2P=H22+PDF*H23-4.0*H24                                           HDER
      H2M=H22-PDF*H23-4.0*H24                                           HDER
      H3P=H32+PDF*H33+4.0*H34                                           HDER
      H3M=H32+PDF*H33-4.0*H34                                           HDER
      RETURN                                                            HDER
      END                                                               HDER
```

```
      SUBROUTINE TERM3(H23,H33)
      DIMENSION SM(128),FCT2(8,128),FCT3(8,128)
      DATA SM/4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2./
      TCCN=0.01
      PD2=1.5707963
      DIST=PD2
      CALL SRTGK(0.0,FL2,FL3)
      CALL SRTGK(PD2,FH2,FH3)
      EFCN2 = FL2+FH2
      EFCN3 = FL3+FH3
      NREP=0
      NINT=1
   20 IF(NINT.GE.128)  GO TO 30
      NREP=NREP+1
      NINT=NINT+NINT
      SUM2=EFCN2
      SUM3=EFCN3
      NORD=NINT-1
      S=DIST/FLOAT(NINT)
      DO 10 N=1,NORD
      NTEST=N-(N/2)*2
      ADD=FLOAT(N)*S
      IF(NTEST.EQ.1)  GO TO 40
      ND2=N/2
      FCT2(NREP,N)=FCT2(NREP-1,ND2)
      FCT3(NREP,N)=FCT3(NREP-1,ND2)
      GO TO 50
   40 CALL SRTGK(ADD,FCT2(NREP,N),FCT3(NREP,N))
   50 CONTINUE
```

```
      SUM2=SUM2+SM(N)*FCT2(NREP,N)                                      TRM3  37
      SUM3=SUM3+SM(N)*FCT3(NREP,N)                                      TRM3  38
   10 CONTINUE                                                          TRM3  39
      IF(NINT.LT.4) GO TO 20                                            TRM3  40
      IF(NINT.EQ.4)  TEST22=SUM2/FLOAT(NINT)                            TRM3  41
      IF(NINT.EQ.4)  TEST33=SUM3/FLOAT(NINT)                            TRM3  42
      TEST21=TEST22                                                     TRM3  43
      TEST31=TEST32                                                     TRM3  44
      IF(NINT.EQ.4)  GO TO 20                                           TRM3  45
      TEST22=SUM2/FLOAT(NINT)                                           TRM3  46
      TEST32=SUM3/FLOAT(NINT)                                           TRM3  47
      TF2=TCON*ABS(TEST22)                                              TRM3  48
      TF3=TCCN*ABS(TEST32)                                              TRM3  49
      IF(TEST22.EQ.0.0) TF2=TCON*ABS(TEST21)                            TRM3  50
      IF(TEST32.EQ.0.0) TF3=TCON*ABS(TEST31)                            TRM3  51
      IF(ABS(TEST21-TEST22).GT.TF2) GO TO 20                            TRM3  52
      IF(ABS(TEST31-TEST32).GT.TF3) GO TO 20                            TRM3  53
      H23=DIST*TEST22/3.0                                               TRM3  54
      H33=DIST*TEST32/3.0                                               TRM3  55
      RETURN                                                            TRM3  56
   30 CONTINUE                                                          TRM3  57
C     WRITE(6,1) TEST21,TEST22                                          TRM3  58
C     WRITE(6,1) TEST31,TEST32                                          TRM3  59
      H23=DIST*TEST22/3.0                                               TRM3  60
      H33=DIST*TEST32/3.0                                               TRM3  61
    1 FORMAT(24H TERM3 DOES NOT CONVERGE,2E16.8)                        TRM3  62
      RETURN                                                            TRM3  63
      END                                                               TRM3  64
```

```
      SUBROUTINE SRTGK(T,F2,F3)
C
C     PURPOSE
C         TO PERFORM K-INTEGRATION FOR THE THIRD TERM
C
C     PARAMETERS
C         T-ANGLE
C
C     COMMON/ITK/
C         GNU-NU
C         H-DEPTH OF WATER
C
      COMMON/THTM/COST,COSST,CSTXMA
      COMMON/CITG/H,B,Y,YPH,BPH,XMA,GNU,GNUH
      TCON=0.01
C
C     WE NOW FIND K ZERO
C
      COST=COS(T)
      COSST=COST*COST
      CSTXMA=COST*XMA
      DIFTST=0.0001/H
      CONST=COSST/GNU
      IF(CONST.GT.H) GO TO 7
      CK2=1.0/CONST
      CK1=CK2*0.75
      THCK2=TANH(H*CK2)-0.9999
      IF(THCK2)  11,12,12
   11 DO 10 N=1,15
      TESTDF=CK2-CK1
      IF(TESTDF.LT.DIFTST)   GO TO 15
      THCK1= TANH(H*CK1)
      TEST1=CONST*CK1-THCK1
      IF(TEST1)14,15,13
   13 CK2=CK1
      CK1=CK2-TESTDF
```

```
      GO TO 10                                                          STGK  37
   14 CK1=(CK1+CK2)/2.0                                                 STGK  38
   10 CONTINUE                                                          STGK  39
   12 CKZERO=CK2                                                        STGK  40
      GO TO 6                                                           STGK  41
    7 CKZERO=0.0                                                        STGK  42
      GO TO 6                                                           STGK  43
   15 CKZERO=CK1                                                        STGK  44
    6 CONTINUE                                                          STGK  45
      SUM2=0.0                                                          STGK  46
      SUM3=0.0                                                          STGK  47
      IF(CKZERO.GT.1.5)  GO TO 100                                      STGK  48
C                                                                       STGK  49
C     INTEGRATION OVER SINGULARITY FOR SMALL K-ZERO                     STGK  50
C                                                                       STGK  51
      ENDOUT=2.0*CKZERO                                                 STGK  52
      CALL CRANK(0.0,ENDOUT,A2,A3)                                      STGK  53
      SUM2=SUM2+A2                                                      STGK  54
      SUM3=SUM3+A3                                                      STGK  55
      STST2=SUM2                                                        STGK  56
      STST3=SUM3                                                        STGK  57
      GO TO 220                                                         STGK  58
  100 IF(CKZERO.GT.3.0)  GO TO 110                                      STGK  59
C                                                                       STGK  60
C     INTEGRATION OVER SINGULARITY FOR MEDIUM K-ZERO                    STGK  61
C                                                                       STGK  62
      ENDLOW=0.0                                                        STGK  63
      ENDIN=0.6667*CKZERO                                               STGK  64
      ENDOUT=CKZERO+CKZERO-ENDIN                                        STGK  65
      CALL SCRANK(0.0,ENDIN,A2,A3)                                      STGK  66
      SUM2=SUM2+A2                                                      STGK  67
      SUM3=SUM3+A3                                                      STGK  68
      STST2=SUM2                                                        STGK  69
      STST3=SUM3                                                        STGK  70
      CALL CRANK(ENDIN,ENDOUT,A2,A3)                                    STGK  71
      SUM2=SUM2+A2                                                      STGK  72
```

```
      SUM3=SUM3+A3
      GO TO 200
C
C     INTEGRATION OVER SINGULARITY FOR LARGE K-ZERO
C
  110 ENDIN=CKZERO-1.0
      ENDOUT=CKZERO+1.0
      INCOEF=ENDIN/1.5
  400 FORMAT(I10,E16.8)
      ICF=INCOEF-1
      COEFF=ENDIN/FLOAT(INCOEF)
      LIMIT=INCOEF
      IF(INCOEF.GT.40)  LIMIT=40
      DO 120 N=1,LIMIT
      ENDLOW=(FLOAT(N)-1.0)*COEFF
      ENDHI=ENDLOW+COEFF
      IF(N.EQ.INCOEF)  ENDHI=ENDIN
      CALL SCRANK(ENDLOW,ENDHI,A2,A3)
      SUM2=SUM2+A2
      SUM3=SUM3+A3
      IF(N.EQ.1)  GO TO 150
      TF2=TCON*ABS(SUM2)
      TF3=TCON*ABS(SUM3)
      IF(SUM2.EQ.0.0) TF2=TCON*ABS(STST2)
      IF(SUM3.EQ.0.0) TF3=TCON*ABS(STST3)
      IF(ABS(SUM2-STST2).LE.TF2.AND.ABS(SUM3-STST3).LE.TF3) GO TO 140
C     IF(N.EQ.40) WRITE(6,405) SUM2,STST2,SUM3,STST3
  405 FORMAT(42HWARNING-LOOP 120 INTGK DOES NOT CONVERGE ,4E16.8)
  150 STST2=SUM2
      STST3=SUM3
  120 CONTINUE
      CALL CRANK(ENDIN,ENDOUT,A2,A3)
      SUM2=SUM2+A2
      SUM3=SUM3+A3
  200 CONTINUE
      TF2=TCON*ABS(SUM2)
```

```
      TF3=TCCN*ABS(SUM3)
      IF(SUM2.EQ.0.0) TF2=TCON*ABS(STST2)
      IF(SUM3.EQ.0.0) TF3=TCON*ABS(STST3)
      IF(ABS(SUM2-STST2).LE.TF2.AND.ABS(SUM3-STST3).LE.TF3) GO TO 140
      STST2=SUM2
      STST3=SUM3
  220 CONTINUE
C
C     INTEGRATION TO INFINITY
C
      DO 130  N=1, 5
      ENDIN=ENDOUT
      ENDOUT=ENDOUT+2.0
      CALL SCRANK(ENDIN,ENDOUT,A2,A3)
      SUM2=SUM2+A2
      SUM3=SUM3+A3
      TF2=TCON*ABS(SUM2)
      TF3=TCON*ABS(SUM3)
      IF(SUM2.EQ.0.0) TF2=TCON*ABS(STST2)
      IF(SUM3.EQ.0.0) TF3=TCON*ABS(STST3)
      IF(ABS(SUM2-STST2).LE.TF2.AND.ABS(SUM3-STST3).LE.TF3) GC TO 140
      IF(N.EQ. 5)  GO TO 300
      STST2=SUM2
      STST3=SUM3
  130 CONTINUE
  140 F2=SUM2
      F3=SUM3
      RETURN
  300 CONTINUE
C 300 WRITE(6,305) STST2,SUM2,STST3,SUM3
  305 FORMAT(24H INTGK DOES NOT CONVERGE,4E16.8)
      F2=SUM2
      F3=SUM3
      RETURN
      END
```

```
      SUBROUTINE CRANK(L,H,F2,F3)                                       CRNK   1
      REAL L                                                            CRNK   2
      TCON=0.01                                                         CRNK   3
      NORD=2                                                            CRNK   4
      CALL  SRTEGK(L,A2,A3)                                             CRNK   5
      CALL  SRTEGK(H,B2,B3)                                             CRNK   6
      SUM2=(A2+B2)*0.5                                                  CRNK   7
      SUM3=(A3+B3)*0.5                                                  CRNK   8
   20 IF(NORD.GT.100)  GO TO 30                                         CRNK   9
      NORD=3*NORD-2                                                     CRNK  10
      NSP=NORD-1                                                        CRNK  11
      NREP=3*(NORD/2)-2                                                 CRNK  12
      D=H-L                                                             CRNK  13
      A=D/FLOAT(NSP)/3.0                                                CRNK  14
      DO 10 N=1,NREP                                                    CRNK  15
      NTEST=N-(N/3)*3                                                   CRNK  16
      IF(NTEST.EQ.0)  GO TO 10                                          CRNK  17
      B=A*FLOAT(N)                                                      CRNK  18
      CALL  SRTEGK(L+B,A2,A3)                                           CRNK  19
      CALL  SRTEGK(H-B,B2,B3)                                           CRNK  20
      SUM2=(A2+B2)+SUM2                                                 CRNK  21
      SUM3=(A3+B3)+SUM3                                                 CRNK  22
   10 CONTINUE                                                          CRNK  23
      IF(NORD.LT.10) GO TO 20                                           CRNK  24
      IF(NORD.LT.11) TEST22=SUM2/FLOAT(NSP)                             CRNK  25
      IF(NORD.LT.11) TEST23=SUM3/FLOAT(NSP)                             CRNK  26
      TEST12=TEST22                                                     CRNK  27
      TEST13=TEST23                                                     CRNK  28
      IF(NORD.LT.11) GO TO 20                                           CRNK  29
      TEST22=SUM2/FLOAT(NSP)                                            CRNK  30
      TEST23=SUM3/FLOAT(NSP)                                            CRNK  31
      TF2=TCON*ABS(TEST22)                                              CRNK  32
      TF3=TCCN*ABS(TEST23)                                              CRNK  33
      IF(TEST22.EQ.0.0)  TF2=TCON*ABS(TEST12)                           CRNK  34
      IF(TEST23.EQ.0.0)  TF3=TCON*ABS(TEST13)                           CRNK  35
      IF(ABS(TEST12-TEST22).GT.TF2) GO TO 20                            CRNK  36
```

```
      IF(ABS(TEST13-TEST23).GT.TF3) GO TO 20                            CRNK  37
      F2=A*SUM2                                                         CRNK  38
      F3=A*SUM3                                                         CRNK  39
      RETURN                                                            CRNK  40
   30 CONTINUE                                                          CRNK  41
C     WRITE(6,1) TEST12,TEST22,TEST13,TEST23                            CRNK  42
    1 FORMAT(32H CRANK DOES NOT CONVERGE. TEST1=,E16.8,2X               CRNK  43
     1,6HTEST2=,E16.8)                                                  CRNK  44
      F2=A*SUM2                                                         CRNK  45
      F3=A*SUM3                                                         CRNK  46
      RETURN                                                            CRNK  47
      END                                                               CRNK  48
```

```
      SUBROUTINE SCRANK(L,H,F2,F3)
      REAL L
      DIMENSION SM(256),FCT2(10,256),FCT3(10,256)
      DATA SM/4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2./
      TCON=0.01
      DIST=H-L
      CALL SRTEGK(L,A2,A3)
      CALL SRTEGK(H,B2,B3)
      EFCN2=A2+B2
      EFCN3=A3+B3
      NREP=0
      NINT=1
   20 IF(NINT.GE.128)  GO TO 30
      NREP=NREP+1
      NINT=NINT+NINT
      SUM2=EFCN2
      SUM3=EFCN3
      NORD=NINT-1
      S=DIST/FLOAT(NINT)
      DO 10  N=1,NORD
      NTEST=N-(N/2)*2
```

```
      ADD=FLOAT(N)*S                                                    SRNK  37
      IF(NTEST.EQ.1)  GO TO 40                                          SRNK  38
      ND2=N/2                                                           SRNK  39
      FCT2(NREP,N)=FCT2(NREP-1,ND2)                                     SRNK  40
      FCT3(NREP,N)=FCT3(NREP-1,ND2)                                     SRNK  41
      GO TO 50                                                          SRNK  42
   40 CALL SRTEGK(L+ADD,FCT2(NREP,N),FCT3(NREP,N))                      SRNK  43
   50 CONTINUE                                                          SRNK  44
      SUM2=SUM2+SM(N)*FCT2(NREP,N)                                      SRNK  45
      SUM3=SUM3+SM(N)*FCT3(NREP,N)                                      SRNK  46
   10 CONTINUE                                                          SRNK  47
      IF(NINT.LT.4)  GO TO 20                                           SRNK  48
      IF(NINT.EQ.4) TEST22=SUM2/FLOAT(NINT)                             SRNK  49
      IF(NINT.EQ.4) TEST23=SUM3/FLOAT(NINT)                             SRNK  50
      TEST12=TEST22                                                     SRNK  51
      TEST13=TEST23                                                     SRNK  52
      IF(NINT.EQ.4)  GO TO 20                                           SRNK  53
      TEST22=SUM2/FLOAT(NINT)                                           SRNK  54
      TEST23=SUM3/FLOAT(NINT)                                           SRNK  55
      TF2=TCON*ABS(TEST22)                                              SRNK  56
      TF3=TCON*ABS(TEST23)                                              SRNK  57
      IF(TEST22.EQ.0.0) TF2=TCON*ABS(TEST12)                            SRNK  58
      IF(TEST23.EQ.0.0) TF3=TCON*ABS(TEST13)                            SRNK  59
      IF(ABS(TEST12-TEST22).GT.TF2) GO TO 20                            SRNK  60
      IF(ABS(TEST13-TEST23).GT.TF3) GO TO 20                            SRNK  61
      F2=DIST*TEST22/3.0                                                SRNK  62
      F3=DIST*TEST23/3.0                                                SRNK  63
      RETURN                                                            SRNK  64
   30 CONTINUE                                                          SRNK  65
C     WRITE(6,1) TEST12,TEST22,TEST13,TEST23                            SRNK  66
    1 FORMAT(25H SCRANK DOES NOT CONVERGE,4E16.8)                       SRNK  67
      F2=DIST*TEST22/3.0                                                SRNK  68
      F3=DIST*TEST23/3.0                                                SRNK  69
      RETURN                                                            SRNK  70
      END                                                               SRNK  71
```

```
      SUBROUTINE SRTEGK(CK,F2,F3)
C
C     PURPOSE
C        TO CALCULATE INTEGRAND FOR THE THIRD TERM
C
      COMMON/CITG/H,B,Y,YPH,BPH,XMA,GNU,GNUH
      COMMON/THTM/COST,COSST,CSTXMA
      PI=3.1415927
      CKH=CK*H
      IF(CK.LT.0.0001)  GO TO 200
      IF((CK*(Y+B)).LT.-150.0.AND.(CK*(Y-H)).LT.-150.0)  GO TO 200
      IF((CK*(Y+B)).LE.-30.0.AND.(CK*(Y-H)).LE.-30.0)  GO TO 110
      IF(CKH.GT.30.0)  GO TO 110
  140 EXPKH=EXP(CKH)
      DEXPKH=1.0/EXPKH
      EXPKYH=EXP(CK*YPH)
      SHKYH=(EXPKYH-1.0/EXPKYH)/2.0
      EXPKBH=EXP(CK*BPH)
      CHBYH=(EXPKBH+1.0/EXPKBH)/2.0
      PROD=CHBYH*(CK*COSST+GNU)-GNU
      TERM=COS(CK*CSTXMA)
      TER2=SIN(CK*CSTXMA)
      DENOM=(CK*COSST*(EXPKH+DEXPKH)-GNU*(EXPKH-DEXPKH))/2.0
      TT=CK*CK*COST*DEXPKH*PROD*SHKYH/DENOM
      F2=TT*TER2
      F3=TT*CK*COST*TERM
      RETURN
  110 CONTINUE
      DEXPKH=EXP(-CKH)
      EXPKY=EXP(CK*Y)
      EXPKB=EXP(CK*B)
      DENOM=CK*COSST-GNU
      FACTOR=(CK*COSST+GNU)
      ARG=CK*XMA*COST
      PROD=COS(ARG)
      PR2=SIN(ARG)
```

```
      TT=CK*CK*COST*EXPKY*(EXPKB*FACTOR-2.0*GNU*DEXPKH)/DENCM/2.0
      F2=TT*PR2
      F3=TT*CK*COST*PROD
      RETURN
  200 F2=0.0
      F3=0.0
      RETURN
      END
```

```
      SUBROUTINE TERM4(E,H24,H34)
      DIMENSION SM(64),FCT2(7,64),FCT3(7,64)
      DATA SM/4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.
     1,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2.,4.,2./
      TCON=0.01
      PD2=1.5707963
      DIST=PD2-E
      CALL SRTEG(E,FL2,FL3)
      CALL SRTEG(PD2,FH2,FH3)
      EFN2=FL2+FH2
      EFN3=FL3+FH3
      NREP=0
      NINT=1
   20 IF(NINT.GE.64)  GO TO 30
      NREP=NREP+1
      NINT=NINT+NINT
      SUM2=EFN2
      SUM3=EFN3
      NORD=NINT-1
      S=DIST/FLOAT(NINT)
      DO 10 N=1,NORD
      NTEST=N-(N/2)*2
      ADD=FLOAT(N)*S
      IF(NTEST.EQ.1)  GO TO 40
      ND2=N/2
      FCT2(NREP,N)=FCT2(NREP-1,ND2)
      FCT3(NREP,N)=FCT3(NREP-1,ND2)
      GO TO 45
   40 CALL SRTEG(E+ADD,FCT2(NREP,N),FCT3(NREP,N))
   45 CONTINUE
      SUM3=SUM3+FCT3(NREP,N)*SM(N)
      SUM2=SUM2+FCT2(NREP,N)*SM(N)
   10 CONTINUE
      IF(NINT.LT.4) GO TC 20
```

```
      IF(NINT.EQ.4) TEST22=SUM2/FLOAT(NINT)
      IF(NINT.EQ.4) TEST23=SUM3/FLOAT(NINT)
      TEST12=TEST22
      TEST13=TEST23
      IF(NINT.EQ.4)  GO TO 20
      TEST22=SUM2/FLOAT(NINT)
      TEST23=SUM3/FLOAT(NINT)
      TF2=TCON*ABS(TEST22)
      TF3=TCON*ABS(TEST23)
      IF(TEST22.EQ.0.0) TF2=TCON*ABS(TEST12)
      IF(TEST23.EQ.0.0) TF3=TCON*ABS(TEST13)
      IF(ABS(TEST12-TEST22).GT.TF2) GO TO 20
      IF(ABS(TEST13-TEST23).GT.TF3) GO TO 20
      H24=DIST*TEST22/3.0
      H34=DIST*TEST23/3.0
      RETURN
   30 CONTINUE
C     WRITE(6,1) TEST12,TEST22
C     WRITE(6,1) TEST13,TEST23
      H24=DIST*TEST22/3.0
      H34=DIST*TEST23/3.0
    1 FORMAT(24H TERM4 DOES NOT CONVERGE,2E16.8)
      RETURN
   50 FORMAT(7H E,DIST,2E16.8)
   55 FORMAT(5H SUM ,E16.8)
   60 FORMAT(12H NREP,N,FCT ,2I5,E16.8)
   65 FORMAT(7H TERM4 ,E16.8)
      END
```

```
      SUBROUTINE SRTEG(T,F2,F3)
      COMMON/CITG/H,B,Y,YPH,BPH,XMA,GNU,GNUH
      REAL INTE1,INTE2
      PI=3.1415927
C
C     WE NOW FIND K ZERO
      COST=COS(T)
      COSST=COST*COST
      DIFTST=0.0001/H
      CONST=COSST/GNU
      IF(CONST.GT.H)  GO TO 7
      CK2=1.0/CONST
      CK1=CK2*0.75
      THCK2=TANH(H*CK2)-0.9999
      IF(THCK2)  11,12,12
   11 DO 10 N=1,15
      TESTDF=CK2-CK1
      IF(TESTDF.LT.DIFTST)  GO TO 15
      THCK1= TANH(H*CK1)
      TEST1=CONST*CK1-THCK1
      IF(TEST1)14,15,13
   13 CK2=CK1
      CK1=CK2-TESTDF
      GO TO 10
   14 CK1=(CK1+CK2)/2.0
   10 CONTINUE
   12 CKZERO=CK2
      GO TO 6
    7 CKZERO=0.0
      GO TO 6
   15 CKZERO=CK1
    6 CONTINUE
      CKH=CKZERO*H
      BPY=B+Y
C
C     TEST FOR MAGNITUDE OF K
```

```
C                                                                       STEG  37
      IF(CKZERO.LT.0.0001)  GO TO 250                                   STEG  38
      IF(CKZERO.GT.150.0)  GO TO 250                                    STEG  39
      IF((CKZERO*BPY).LT.-175.0.AND.(CKZERO*(Y-H)).LT.-175.0)  GO TO 250STEG  40
      IF((CKZERO*BPY).LT.-30.0.AND.(CKZERO*(Y-H)).LE.-30.) GO TO 600    STEG  41
      IF(CKH.GT.30.0)  GO TO 600                                        STEG  42
C                                                                       STEG  43
C     CALCULATE INTEGRAND FOR SMALL K                                   STEG  44
C                                                                       STEG  45
      EXPKH=EXP(CKH)                                                    STEG  46
      DEXPKH=1.0/EXPKH                                                  STEG  47
      SECHKH=2.0/(EXPKH+DEXPKH)                                         STEG  48
      SHKHS=SECHKH*SECHKH                                               STEG  49
      EXPKYH=EXP(CKZERO*YPH)                                            STEG  50
      SIHKYH=(EXPKYH-1.0/EXPKYH)/2.0                                    STEG  51
      EXPKBH=EXP(CKZERO*BPH)                                            STEG  52
      PAREN=(EXPKBH+1.0/EXPKBH)/2.0*(CKZERO*COSST+GNU)-GNU              STEG  53
      CTKXMA=CKZERO*COST*XMA                                            STEG  54
      PROD=SIN(CTKXMA)                                                  STEG  55
      PR2=COS(CTKXMA)                                                   STEG  56
      DENOM=COSST-GNUH*SHKHS                                            STEG  57
      TT=CKZERO*CKZERO*COST*DEXPKH*SECHKH*SIHKYH*PAREN/DENOM            STEG  58
      F2=TT*PR2                                                         STEG  59
      F3=TT*CKZERO*COST*PROD                                            STEG  60
      RETURN                                                            STEG  61
  600 CONTINUE                                                          STEG  62
C                                                                       STEG  63
C     CALCULATE INTEGRAND FOR LARGE K                                   STEG  64
C                                                                       STEG  65
      IF(CKF.GT.090.0)  GO TO 610                                       STEG  66
      DEXPKH=EXP(-CKH)                                                  STEG  67
      GO TO 620                                                         STEG  68
  610 DEXPKH=0.0                                                        STEG  69
  620 CONTINUE                                                          STEG  70
      EXPKY=EXP(CKZERO*Y)                                               STEG  71
      EXPKB=EXP(CKZERO*B)                                               STEG  72
```

```
      TERM=CKZERO*COSST+GNU                                                 STEG  73
      DENOM=COSST-4.0*GNUH*DEXPKH*DEXPKH                                    STEG  74
      ARG=CKZERO*XMA*COST                                                   STEG  75
      PROD=SIN(ARG)                                                         STEG  76
      PR2=CCS(ARG)                                                          STEG  77
      TT=CKZERO*CKZERO*COST*EXPKY*(EXPKB*TERM-2.0*GNU*DEXPKH)               STEG  78
     1/DENOM/2.0                                                            STEG  79
      F2=TT*PR2                                                             STEG  80
      F3=TT*CKZERO*COST*PROD                                                STEG  81
      RETURN                                                                STEG  82
  250 F2=0.0                                                                STEG  83
      F3=0.0                                                                STEG  84
      RETURN                                                                STEG  85
      END                                                                   STEG  86
```